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A model of Brownian motion in an inhomogeneous environment

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Abstract

The stochastic motion of a wall of mass M separating two semi-infinite cylindrical volumes filled with non-interacting point particles of mass m is studied. The initial equilibrium states on both sides are characterized by the same pressure but by different temperatures and densities. Frictionless motion of the wall is entirely due to collisions. In the scaling limit where the mass M grows as the surface area of the wall $M \sim L^2, L \to \infty$ and $\epsilon = 2m/(m + M) \sim L^{-2}$, the Ornstein–Uhlenbeck process is shown to govern the stochastic dynamics. In an inhomogeneous environment the wall behaves like a Brownian particle with an effective temperature determined by the ratio of the energy and particle fluxes. The global particle flux defines the dynamic friction coefficient. A comparison with predictions of Boltzmann's theory is presented, bringing in the subtle problem of the drift velocity.

1. Introduction

The relaxation of the velocity distribution of a Brownian particle immersed in a gas at thermodynamic equilibrium is known to follow the Ornstein–Uhlenbeck process [1] (see also [2] and references therein). This Gaussian, stationary and Markovian stochastic process transforms the initial state $\Phi(\boldsymbol{V}; 0) = \delta(\boldsymbol{V})$ (Brownian particle at rest) into the equilibrium Maxwell distribution according to the formula

$$\Phi(\boldsymbol{V}; t) = \left\{\frac{M}{2\pi k_B T[1 - \exp(-2\gamma t)]}\right\}^{3/2} \exp\left\{-\frac{MV^2}{2k_B T[1 - \exp(-2\gamma t)]}\right\}, \quad (1)$$

where $t > 0$ denotes time, $V = |\boldsymbol{V}|$, M is the mass of the Brownian particle, γ is the friction coefficient and k_B and T are Boltzmann's constant and the temperature of the gas, respectively. The evolution law (1) is entirely induced by collisions.

Our object is to study a more general situation, where the surrounding gas is not at thermal equilibrium because of the presence of spatial inhomogeneities in its density and temperature fields. To this end we have chosen a particularly simple model inspired by the still challenging *adiabatic piston problem* [3]. The kind of analysis presented here follows the ideas of Sinai [4], indicating the method for a mathematically rigorous study of the stochastic dynamics. Some previous studies of self-diffusion in inhomogeneous media can be found in [5, 6]. The case of the Brownian particle interacting with two different gas mixtures having the same temperatures, pressures and densities has been studied in [7], with the idea of going beyond the linear Fokker–Planck equation.

Consider an ideal gas enclosed in an infinite three-dimensional cylinder with a cross section area L^2. A mobile flat wall perpendicular to the symmetry axis of the cylinder partitions the gas into two disjoint semi-infinite volumes. The wall has a constant surface mass density σ, so its total mass M is given by

$$M = \sigma L^2. \tag{2}$$

We choose the coordinate system whose X-axis coincides with the symmetry axis of the cylinder.

In the present study the wall with its single degree of freedom (translations along the X-axis) will play the role of the massive Brownian particle. We denote by $X(t)$ and $V(t)$ its position and velocity, respectively. The motion of the wall results from collisions with point particles of mass m constituting the surrounding gas. Specular reflections of these particles at the immobile lateral walls of the cylinder do not change X-components of their velocities. With no loss of generality we shall thus suppose that all velocities are oriented along the X-axis. This property remains conserved under collisions with the wall.

In the present paper we suppose that at the initial moment $t = 0$ the volumes of the gas on the left- and on the right-hand side of the wall are in homogeneous states with different (in general) number densities and temperatures, but with the same pressures. More precisely, at $t = 0$,

(i) the wall is at rest at the origin, $X(0) = 0$, $V(0) = 0$,

(ii) the gases on the left- and on the right-hand side of the wall have Poisson's spatial distributions with parameters (number densities) ρ^- and ρ^+, respectively,

(iii) the probability density for finding a particle of the gas with velocity v is given by $\phi^-(v) = \phi^-(-v)$ within the left subvolume, and by $\phi^+(v) = \phi^+(-v)$ within the right subvolume, the velocities of different particles being uncorrelated,

(iv) $\phi^-(v) = 0$ outside the interval $0 < v^-_{min} < |v| < v^-_{max}$

and similarly

$$\phi^+(v) = 0 \qquad \text{outside the interval } 0 < v^+_{min} < |v| < v^+_{max},$$

(v) the pressures within the two subvolumes are equal

$$\rho^- \int \mathrm{d}v\, mv^2\phi^-(v) = \rho^- k_B T^- = \rho^+ k_B T^+ = \rho^+ \int \mathrm{d}v\, mv^2\phi^+(v)$$

but the densities ρ^-, ρ^+ and the temperatures T^-, T^+ are in general different (inhomogeneity).

The aim of our study is to determine the nature of the stochastic process followed by the velocity $V(t)$ of the moving Brownian wall in the asymptotic scaling region $L \to \infty$, where

$$\frac{m}{M} = \frac{m}{\sigma L^2} \ll 1. \tag{3}$$

2. Constructing basic stochastic fields

Consider the division of the time axis into intervals of equal length Δ. Suppose that we know the velocity of the wall at time $t_n = n\Delta$. We denote it by

$$V_n = V(t_n) = V(n\Delta). \tag{4}$$

The value of V_{n+1} then follows from the binary collision law. When a gas particle with velocity v hits the wall moving with velocity V, the velocity of the wall after collision is given by

$$V' = V - \frac{2m}{m + M}(V - v) = (1 - \epsilon)V + \epsilon v, \tag{5}$$

where

$$\epsilon = \frac{2m}{(m + \sigma L^2)} \ll 1. \tag{6}$$

Suppose that within the time interval (t_n, t_{n+1}) the wall suffers k_n collisions at the moments

$$t_n + \sum_{r=1}^{j} \tau_r, \qquad j = 1, 2, \ldots, k_n \qquad \text{where } \tau_r > 0, r = 1, \ldots, k_n, \ \sum_{r=1}^{k_n} \tau_r < \Delta. \tag{7}$$

We denote by v_j the precollisional velocity of the gas particle at the jth encounter. By iterating the collision law (5) k_n times we arrive at the exact relation

$$V_{n+1} = (1 - \epsilon)^{k_n} V_n + \epsilon \sum_{j=1}^{k_n} (1 - \epsilon)^{k_n - j} v_j. \tag{8}$$

In the asymptotic region (6) we put

$$(1 - \epsilon)^{k_n} = \exp[k_n \log(1 - \epsilon)] \approx \exp(-\epsilon k_n), \tag{9}$$

and similarly

$$(1 - \epsilon)^{k_n - j} \approx \exp[-\epsilon(k_n - j)]. \tag{10}$$

The asymptotic form of equation (8) thus reads

$$V_{n+1} = \exp(-\epsilon k_n) V_n + \epsilon \sum_{j=1}^{k_n} v_j \exp[-\epsilon(k_n - j)]. \tag{11}$$

In equation (11) there appear the following stochastic variables:

(i) k_n = number of collisions during the time interval $[t_n, t_{n+1}]$ and
(ii) v_j = precollisional velocity of the gas particle at the jth encounter, $j = 1, 2, \ldots, k_n$.

The properties of k_n and v_j can be deduced from the fact that the gas particles which collide with the wall within the time interval $[t_n, t_{n+1}] = [t_n, t_n + \Delta]$ belong at $t = 0$ to the subset A of the one-particle phase space defined by

$$A = \{(x, v) : x + vt = X(t), t_n < t < t_n + \Delta\}. \tag{12}$$

If the wall were clamped, there would be no macroscopic (mean) force acting on it owing to the assumed equality of the initial pressures. According to the collision law (5), the encounters with the gas particles introduce changes in $V(t)$ of the order of $\epsilon \ll 1$. An event consisting in building up a velocity of the order $\epsilon^0 = 1$ is thus not likely to occur. Consequently, we assume that in region (6) the collision-induced wall velocity remains small in the course of time

$$|V(t)| < \min\{v_{min}^-, v_{min}^+\}, \tag{13}$$

so recollisions of the wall with the particles which hit it are of vanishing probability.

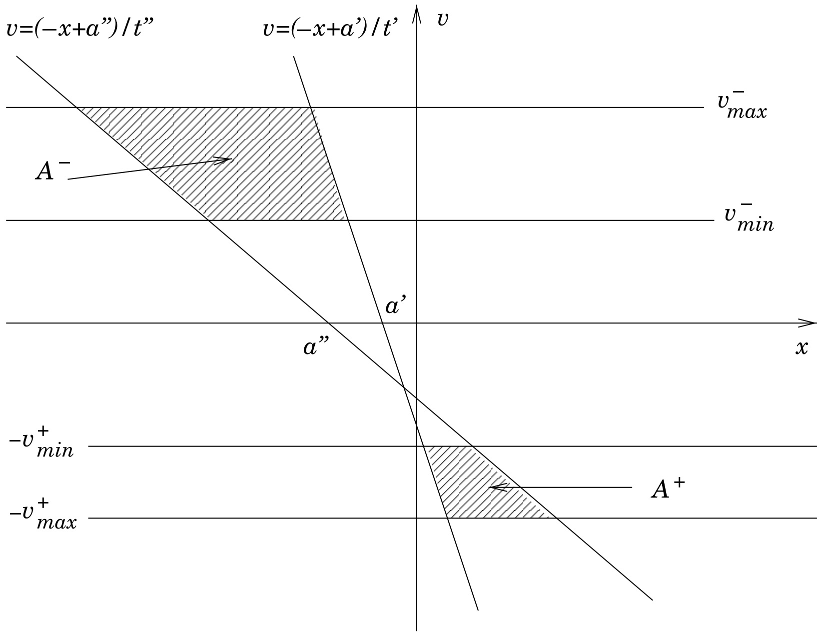

Figure 1. Geometry of the phase space region generating collisions.

Geometrically, (13) implies that the set A is the sum of two trapezoids A^- and A^+, corresponding to the particles arriving from the left- and from the right-hand side of the wall, respectively (see figure 1). Notice that the straight lines with the slopes $(-1/t')$ and $(-1/t'')$ delimiting the trapezoids in figure 1 cross within the region $|v| < \min\{v^-_{min}, v^+_{min}\}$. Assuming $t'' > t'$, we introduce the notation

$$A[t', t''; a', a''] = A^-[t', t''; a', a''] \cup A^+[t', t''; a', a''] \tag{14}$$

where

$$\begin{aligned} A^-[t', t''; a', a''] &= \{(x, v) : a'' - vt'' < x < a' - vt'; v^-_{min} < v < v^-_{max}\} \\ A^+[t', t''; a', a''] &= \{(x, v) : a' - vt' < x < a'' - vt''; v^+_{min} < (-v) < v^+_{max}\}. \end{aligned} \tag{15}$$

Then, the relevant subset (12) of the phase space, from which the colliding particles originate, can be written as $A[t_n, t_{n+1}; X(t_n), X(t_{n+1})]$. The geometrical structure shown in figure 1 permits us to realize the important fact that with the time increasing from t_n to $(t_n + \Delta)$, new regions of the phase space are always added, eventually building the complete set $A[t_n, t_{n+1}; X(t_n), X(t_{n+1})]$. Let us now define the stochastic variables

$$N(t', t''; a', a'') = N^-(t', t''; a', a'') + N^+(t', t''; a', a'') \tag{16}$$

$$\Phi(t', t''; a', a'') = \Phi^-(t', t''; a', a'') + \Phi_+(t', t''; a', a''), \tag{17}$$

where $N^{\mp}(t', t''; a', a'')$ is the number of particles in the set $A^{\mp}[t', t''; a', a'']$, and $\Phi^{\mp}(t', t''; a', a'')$ is the sum of the velocities of all the particles belonging to $A^{\mp}[t', t''; a', a'']$.

The assumed initial condition implies the following properties of the stochastic fields N and Φ.

(1) N^- has the Poisson distribution with parameter $\lambda^- = \langle N^-\rangle = \langle (N^- - \langle N^-\rangle)^2\rangle$ given by

$$\lambda^- = \tfrac{1}{2}\rho^- L^2[\langle |v|\rangle^-(t'' - t') + a' - a''], \tag{18}$$

where $\langle |v|\rangle^- = \int dv\, |v|\phi^-(v)$.

(2) N^+ has the Poisson distribution with parameter $\lambda^+ = \langle N^+\rangle = \langle (N^+ - \langle N^+\rangle)^2\rangle$ given by

$$\lambda^+ = \tfrac{1}{2}\rho^+ L^2[\langle|v|\rangle^+(t'' - t') + a'' - a'], \tag{19}$$

where $\langle|v|\rangle^+ = \int \mathrm{d}v\, |v|\phi^+(v)$.

(3) The field (16) has the Poisson distribution with parameter $\lambda = \lambda^- + \lambda^+$.

The probability density for the field $\Phi^+(t,', t''; a', a'')$ taking the value u is given by

$$\chi^+(u) = \exp(-\lambda^+)\sum_{s=0}^{\infty}\frac{(\rho^+L^2)^s}{s!}\int_{-\infty}^{0}\mathrm{d}v_1\cdots\int_{-\infty}^{0}\mathrm{d}v_s \times \prod_{j=1}^{s}\{\phi^+(v_j)[|v_j|(t''-t') + a'' - a']\}\delta\left(u - \sum_{i=1}^{s} v_i\right). \tag{20}$$

The calculation of the Fourier transform yields

$$\hat{\chi}^+(k) = \int \mathrm{d}u\, \exp(-\mathrm{i}ku)\chi^+(u) = \exp\left\{\frac{\rho^+}{2}L^2\int \mathrm{d}v\,\phi^+(v)(\mathrm{e}^{\mathrm{i}k|v|} - 1)[|v|(t''-t') + a'' - a']\right\}. \tag{21}$$

From equation (21) one readily finds

$$\langle\Phi^+\rangle = -\frac{\rho^+}{2}L^2[\langle v^2\rangle^+(t''-t') + \langle|v|\rangle^+(a''-a')] \tag{22}$$

$$\langle[\Phi^+ - \langle\Phi^+\rangle]^2\rangle = \frac{\rho^+}{2}L^2[\langle|v|^3\rangle^+(t''-t') + \langle v^2\rangle^+(a''-a')]. \tag{23}$$

Along the same lines one can calculate the moments

$$\langle\Phi^-\rangle = \frac{\rho^-}{2}L^2[\langle v^2\rangle^-(t''-t') + \langle|v|\rangle^-(a'-a'')] \tag{24}$$

$$\langle[\Phi^- - \langle\Phi^-\rangle]^2\rangle = \frac{\rho^-}{2}L^2[\langle|v|^3\rangle^-(t''-t') + \langle v^2\rangle^-(a'-a'')]. \tag{25}$$

For the complete field $\Phi = \Phi^- + \Phi^+$ one thus finds

$$\langle\Phi\rangle = \frac{L^2}{2}\big\{[\rho^-\langle v^2\rangle^- - \rho^+\langle v^2\rangle^+](t''-t') + [\rho^-\langle|v|\rangle^- + \rho^+\langle|v|\rangle^+](a'-a'')\big\} \tag{26}$$

$$\langle(\Phi - \langle\Phi\rangle)^2\rangle = \frac{L^2}{2}\big\{[\rho^-\langle|v|^3\rangle^- + \rho^+\langle|v|^3\rangle^+](t''-t') + [\rho^-\langle v^2\rangle^- - \rho^+\langle v^2\rangle^+](a'-a'')\big\}. \tag{27}$$

3. Thermalization of the wall

We now return to the fundamental relation (11). The stochastic variables appearing therein can be expressed in terms of the fields N and Φ. Indeed, by definition

$$k_n = N[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)] \tag{28}$$

and

$$v_j = \sum_{i=1}^{j} v_i - \sum_{i=1}^{j-1} v_i = \Phi\left[t_n, t_n + \sum_{r=1}^{j}\tau_r; X(t_n), X\left(t_n + \sum_{r=1}^{j}\tau_r\right)\right] - \Phi\left[t_n, t_n + \sum_{r=1}^{j-1}\tau_r; X(t_n), X\left(t_n + \sum_{r=1}^{j-1}\tau_r\right)\right]. \tag{29}$$

We begin the study of equation (11) by examining the term $V_n \exp(-\epsilon k_n)$ in the limit $L \to \infty$, assuming the scaling of the time interval

$$\Delta \sim L^{-\gamma}, \qquad 0 < \gamma < 2. \tag{30}$$

To this end we rewrite equation (28) in the form

$$k_n = \langle N[t_n, t_n + \Delta; X(t_n), X(t_n)]\rangle + R_N^1 + R_N^2, \tag{31}$$

where

$$\langle N[t_n, t_n + \Delta; X(t_n), X(t_n)]\rangle = \frac{L^2}{2}[\rho^- \langle |v|\rangle^- + \rho^+ \langle |v|\rangle^+]\Delta \tag{32}$$

(see (16), (18), (19)). The remainder terms R_N^1, R_N^2 are given by

$$R_N^1 = N[t_n, t_n + \Delta; X(t_n), X(t_n)] - \langle N[t_n, t_n + \Delta; X(t_n), X(t_n)]\rangle \tag{33}$$

$$R_N^2 = N[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)] - N[t_n, t_n + \Delta; X(t_n), X(t_n)]. \tag{34}$$

The remainder (33) represents the fluctuation of the Poisson distribution. Using the shorthand notation

$$\alpha = \tfrac{1}{2}[\rho^- \langle |v|\rangle^- + \rho^+ \langle |v|\rangle^+] \tag{35}$$

we thus find

$$\sqrt{\langle (R_N^1)^2\rangle} = L\sqrt{\alpha\Delta}. \tag{36}$$

We shall assume that the mean value (32), equal to $L^2\alpha\Delta$, is dominant in the right-hand side of (31) in the scaling limit under consideration (the proof of this assumption is left as an open problem here). Consequently, we find asymptotically

$$V_n \exp(-\epsilon k_n) \cong V_n \exp(-\epsilon L^2 \alpha\Delta), \tag{37}$$

where, in accordance with definition (6)

$$\lim_{L\to\infty} \epsilon L^2 = \frac{2m}{\sigma}.$$

In equation (11) there appears also the term

$$\epsilon \sum_{j=1}^{k_n} v_j \exp[-\epsilon(k_n - j)] = \epsilon\Phi[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)]$$
$$+ \epsilon(1 - \mathrm{e}^{\epsilon}) \sum_{j=1}^{k_n - 1} \exp[-\epsilon(k_n - j)]\Phi\left[t_n, t_n + \sum_{r=1}^{j}\tau_r; X(t_n), X\left(t_n + \sum_{r=1}^{j}\tau_r\right)\right]. \tag{38}$$

In order to obtain a rough estimate of the contribution containing the factor $\epsilon(1 - \mathrm{e}^{\epsilon})$, we replace the stochastic fields

$$\Phi\left[t_n, t_n + \sum_{r=1}^{j}\tau_r; X(t_n), X\left(t_n + \sum_{r=1}^{j}\tau_r\right)\right], \qquad j = 1, 2, \ldots, (k_n - 1)$$

by

$$\Phi[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)].$$

We then find the expression of the form

$$-\epsilon\{1 - \exp[-\epsilon(k_n - 1)]\}\Phi[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)]$$
$$\cong -\left[\frac{\epsilon(k_n - 1)}{\Delta}\right]\Delta\epsilon\Phi[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)]. \tag{39}$$

According to our estimate of the collision number k_n the ratio $\epsilon(k_n - 1)/\Delta$ is of the order of unity when $L \to \infty$. Moreover, as $\Delta \sim L^{-\gamma}$, $0 < \gamma < 2$, the term (39) becomes asymptotically negligible compared with the first term on the right-hand side of (38). We thus claim that in the scaling limit only the term $\epsilon\Phi[t_n, t_n+\Delta; X(t_n), X(t_n+\Delta)]$ should be retained. According to equation (26), at equal pressures its mean value is given by

$$\langle \epsilon\Phi[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)]\rangle = \epsilon L^2\alpha(X(t_n) - X(t_{n+1}))$$
$$= -\epsilon L^2\alpha \int_{t_n}^{t_n+\Delta} \mathrm{d}\tau\, V(\tau) \approx -\epsilon L^2\alpha\Delta V(t_n). \tag{40}$$

We thus rewrite (40) in the form

$$\epsilon\Phi[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)] = -\epsilon L^2\alpha\Delta V(t_n) + \epsilon\Phi[t_n, t_n + \Delta; X(t_n), X(t_n)]$$
$$+ \epsilon R_\Phi[t_n, t_n + \Delta; X(t_n), X(t_n + \Delta)]. \tag{41}$$

The remainder R_Φ will be assumed here to be negligible with respect to the fluctuating field $\Phi[t_n, t_n + \Delta; X(t_n), X(t_n)]$ (with zero mean value), whose properties can be directly deduced from equations (20)–(27). By retaining only the dominant terms we arrive at the stochastic equation

$$V(t_n + \Delta) = [\exp(-\epsilon L^2\alpha\Delta) - \epsilon L^2\alpha\Delta]V(t_n) + \epsilon\Phi[t_n, t_n + \Delta; X(t_n), X(t_n)]$$
$$\approx \exp(-2\epsilon L^2\alpha\Delta)V(t_n) + \epsilon\Phi[t_n, t_n + \Delta; X(t_n), X(t_n)]. \tag{42}$$

The analysis of the field $\Phi(t', t''; a', a'')$ has shown that the corresponding probability distribution depended on the differences $(t' - t'')$ and $(a' - a'')$ only. So, the probability distribution of $\Phi[t_n, t_n + \Delta; X(t_n), X(t_n)]$ does not depend either on t_n or on $X(t_n)$. In fact, its Fourier transform reads (compare with (21))

$$\hat{\chi}(k) = \exp\left\{ -\frac{L^2\Delta}{2} \int \mathrm{d}v\, |v|[\rho^+\phi^+(v)(1 - \exp(\mathrm{i}k|v|)) + \rho^-\phi^-(v)(1 - \exp(-\mathrm{i}k|v|))]\right\}. \tag{43}$$

Taking into account the equality of pressures (see point (v) in the description of the initial state), we find that in the scaling limit where $L^2\Delta \sim L^{2-\gamma} \to \infty$ the distribution (43) takes the Gaussian form

$$\hat{\chi}(k) \approx \exp\left\{-\frac{L^2}{4}\Delta[\rho^+\langle|v|^3\rangle^+ + \rho^-\langle|v|^3\rangle^-]k^2\right\} = \exp\left(-\frac{L^2}{2}\Delta\delta k^2\right), \tag{44}$$

where

$$\delta = \tfrac{1}{2}[\rho^-\langle|v|^3\rangle^- + \rho^+\langle|v|^3\rangle^+].$$

The field $\Phi[t_n, t_n + \Delta; X(t_n), X(t_n)]$ is thus also asymptotically Gaussian. The corresponding probability density reads

$$\chi(u) = \frac{1}{L\sqrt{2\pi\delta\Delta}} \exp\left(-\frac{u^2}{2L^2\delta\Delta}\right). \tag{45}$$

By iterating the fundamental equation (42) we find the relation

$$V_n = V(t_n) = \epsilon \sum_{j=0}^{n-1} \exp(-2j\alpha\epsilon L^2\Delta)\Phi[t_j, t_j + \Delta; X(t_j), X(t_j)]. \tag{46}$$

The probability distribution governing the velocity of the wall can now be deduced by standard methods (for details see e.g. [1]). Up to a normalizing factor we find

$$\exp\left\{-\frac{V^2}{2L^2\delta\epsilon^2 \sum_{j=0}^{n-1} \Delta \exp(-4j\alpha\epsilon L^2\Delta)}\right\}. \tag{47}$$

In the scaling limit the time interval Δ tends to zero, so choosing a fixed time $n\Delta < t < (n+1)\Delta$ we find that (47) takes the form

$$\exp\left\{-\frac{V^2}{2\delta\epsilon^2 L^2 \int_0^t \mathrm{d}\tau\ \exp(-4\alpha\epsilon L^2\tau)}\right\}. \tag{48}$$

The asymptotic normalized probability density for finding the wall with velocity V at time t reads

$$\chi(V;t) = \sqrt{\frac{2\alpha}{\pi\delta\epsilon[1-\exp(-4\alpha\epsilon L^2 t)]}}\exp\left\{-\frac{2\alpha V^2}{\delta\epsilon[1-\exp(-4\alpha\epsilon L^2 t)]}\right\}. \tag{49}$$

The evolution law (49) is identical to the Ornstein–Uhlenbeck process of thermalization of the Brownian particle starting with zero velocity (compare with (1)), with the dynamical friction coefficient

$$\gamma = 2\alpha\epsilon L^2 = \frac{2m}{\sigma}[\rho^-\langle|v|\rangle^- + \rho^+\langle|v|\rangle^+]. \tag{50}$$

Recalling the definition $\epsilon = 2m/(m+M) \approx 2m/M$ we also readily infer from (49) the temperature T_∞ of the wall attained in the long-time $t \to \infty$ limit

$$k_B T_\infty = \frac{m\delta}{2\alpha} = \frac{m}{2}\left[\frac{\rho^+\langle|v|^3\rangle^+ + \rho^-\langle|v|^3\rangle^-}{\rho^-\langle|v|\rangle^- + \rho^+\langle|v|\rangle^+}\right] \tag{51}$$

(see (35), (45)). T_∞ is thus determined by the ratio of the energy and particle fluxes.

4. Discussion

We based the analysis of the stochastic dynamics of the massive wall on the collision law (5) using the scaling

$$M \sim L^2, \quad \epsilon = \frac{2m}{m+M} \sim L^{-2}, \quad \Delta \sim L^{-\gamma}, \quad 0 < \gamma < 2, \quad L \to \infty. \tag{52}$$

The dynamics has been studied under the equal-initial-pressures assumption. Our derivation sketches the way along which a rigorous derivation of the Ornstein–Uhlenbeck process (49) could be constructed. In particular, the validity of (49) still requires proving that the neglected remainder terms R_N^1, R_N^2, R_Φ (see equations (33), (34), (41)) are indeed vanishingly small compared with those retained in equation (42).

The particles of the ideal gases were treated here as not interacting with each other. It is thus interesting to discover what parameter is responsible in this case for the dynamical friction. According to formula (50) the friction force is simply proportional to the combined particle fluxes arriving from both sides on the wall.

Our approach made the Ornstein–Uhlenbeck process appear in a new case, allowing the spatial inhomogeneity of the system. In order to avoid dealing with effects of recollisions we introduced cut-offs in the velocity spectrum of the initial distributions (13). It is interesting to evaluate the asymptotic temperature (51) of the wall in the case of the Maxwell distributions

$$\phi^\pm(v) = \left\{\frac{m}{2\pi k_B T^\pm}\right\}^{1/2}\exp\left\{-\frac{mv^2}{2k_B T^\pm}\right\} \tag{53}$$

without cut-offs in order to compare our results with those derived within the Boltzmann theory. A straightforward calculation yields in this case

$$T_\infty = \sqrt{T^- T^+}. \tag{54}$$

It turns out that the geometric mean (54) also plays the role of temperature in the dominant Gaussian term of the perturbative solution of the Boltzmann equation corresponding to the stochastic process studied here [8]. This is not surprising, as the fundamental assumption on which Boltzmann's kinetic theory relies is the absence of recollisions. However, the study of the Boltzmann equation has also led to the prediction in the stationary state of a drift velocity induced by the temperature difference and oriented toward the higher-temperature region [8]

$$\lim_{t\to\infty}\langle V(t)\rangle = \frac{\sqrt{2\pi}}{4}\sqrt{\frac{m}{M}}\left[\sqrt{\frac{k_B T^+}{M}} - \sqrt{\frac{k_B T^-}{M}}\right]. \tag{55}$$

The evaluation of the drift (55) required going beyond the dominant term in the perturbative expansion in parameter ϵ. The open question is thus whether it is possible to prove that the remainder terms neglected here contain information about the subtle effect (55). The question is of fundamental interest because it is not clear whether going beyond the Ornstein–Uhlenbeck process makes sense at all, and, if so, how to extend the present approach to determine higher-order corrections. This question requires a detailed study of the remainder terms (33), (34) and (41), which in any case is necessary for constructing a rigorous derivation of the Ornstein–Uhlenbeck process (49). It is to be noted that the existence of the drift velocity has been rigorously proved in the special case of $M = m$, where the mass of the wall was equal to that of the gas particles [9].

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